Razvijanje f-je u Furijeov red u intervalu [a,b], a<b

Neka je $y=f(x)$ integrabilna f-ja na intervalu [a, b]. Brojeve

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx$$

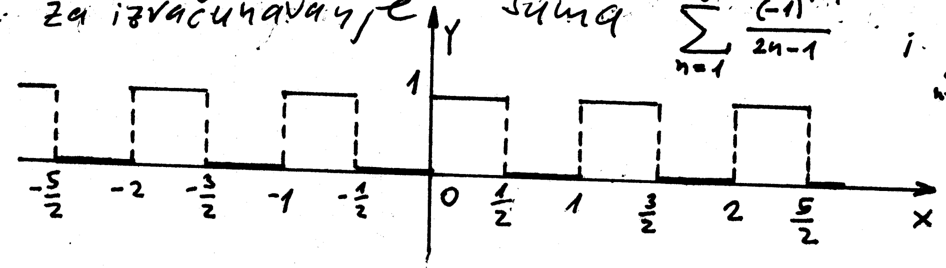
i $$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx, \quad n\in\mathbb{N}$$

nazivamo FURIJEOVI KOEFICIJENTI f-je $f(x)$.
Trigonometrijski red

$$\frac{a_0}{2} + \sum\left(a_n\cos\frac{2n\pi x}{b-a} + b_n\sin\frac{2n\pi x}{b-a}\right), \quad x\in[a,b]$$

se naziva FURIJEOV RED f-je $f(x)$.
Parnost i neparnost možemo ispitati samo u slučaju ako je interval [a, b] simetričan u odnosu na nulu.

(#) Pretvoriti u Furijeov red f-ju definisanu grafikom. Iskoristiti dobijeni rezultat za izračunavanje suma $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ i $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{2n-1}$.



Rj. F-ju predstavljenu grafikom označićemo sa $y=f(x)$.

F-ja je periodična perioda 1, što znači f-ju je dovoljno pretvoriti u Furijeov red u intervalu $[0,1]$.

Furijerovi koeficijenti na intervalu $[a,b]$ se računaju po formuli:

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\quad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx \quad i \quad b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$$

Interval $[0,1]$ nije simetričan u odnosu na 0, pa parnost i neparnost ne igraju nikakvu ulogu.

Furijeov red f-je $f(x)$ je oblika $\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$

$x\in[a,b]$

U našem slučaju:

$$a_0=2\int_0^1 f(x)\,dx=2\int_0^{1/2}dx=2\cdot\frac{1}{2}=1$$

$$a_n=2\int_0^1 f(x)\cos 2n\pi x\,dx=2\int_0^{1/2}\cos 2n\pi x\,dx=2\,\frac{1}{2n\pi}\sin 2n\pi x\Big|_0^{\frac{1}{2}}=\frac{1}{n\pi}\sin n\pi=0$$

$$b_n=2\int_0^1 f(x)\sin 2n\pi x\,dx=2\int_0^{\frac{1}{2}}\sin 2n\pi x\,dx=2\,\frac{1}{2n\pi}\left(-\cos 2n\pi x\Big|_0^{\frac{1}{2}}\right)=\frac{(-1)}{n\pi}(\cos n\pi-1)$$

$$=\frac{(-1)}{n\pi}\left((-1)^n-1\right)=\frac{1+(-1)^{n+1}}{n\pi}$$

$2(2n-1)\pi x$

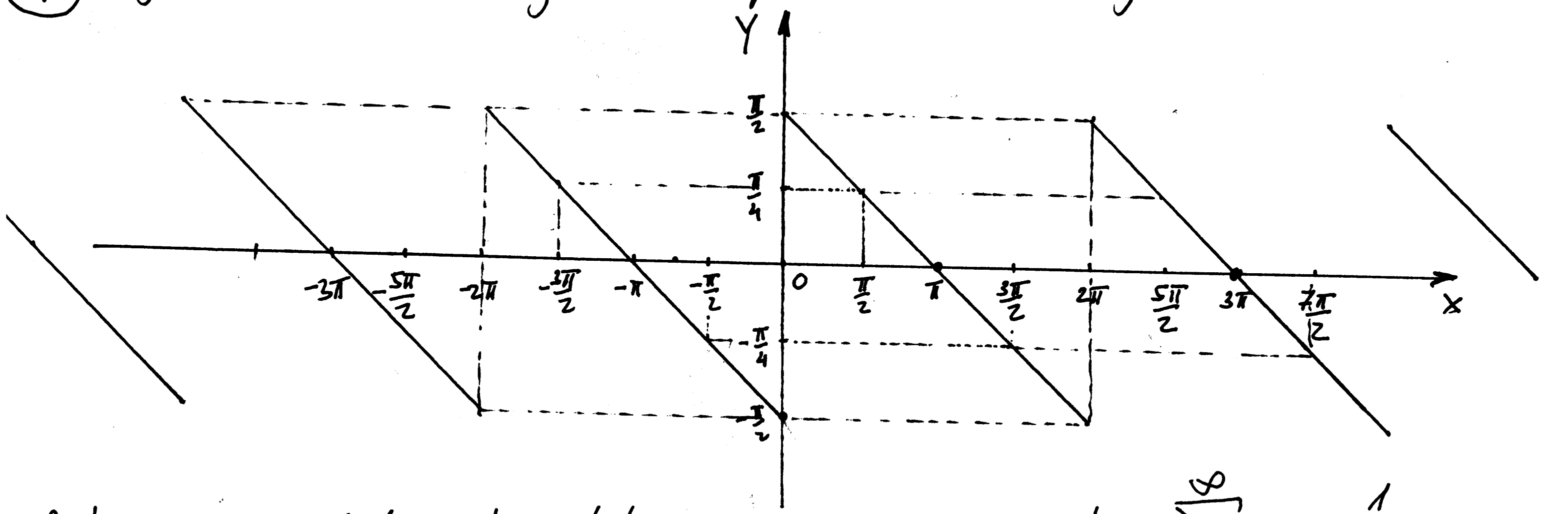
$$f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n\pi}\sin 2n\pi x=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{(2n-1)\pi}\sin 2(2n-1)\pi x$$

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2(2n-1)\pi x}{(2n-1)}$$ f-ja razložena u Fourierov red

$f(\frac{1}{4})=1$ (iz grafika), $\sin 2(2n-1)\pi\cdot\frac{1}{4}=\sin(2n-1)\frac{\pi}{4}=(-1)^{n+1}$ pa je $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}$

$\sum_{n=1}^{\infty}\frac{(-1)^{n}}{2n-1}=-\frac{\pi}{4}$ (ovaj rezultat se može dobiti na dva načina: U Furijeov red uvrstiti tačku $x=\frac{3}{4}$ ili prethodnu sumu pomnožiti sa (-1)).

⑦ F-ju definisanu grafikom pretvoriti u Furijeov red



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{(4n-1)(4n-3)}$.

Rj. Prvo primjetimo da je f-ja periodična što znači da se može pretvoriti u Furijeov red. Dalje, primjetimo da je period 2π što znači da možemo posmatrati npr. interval $[0,2\pi]$.

F-ja na intervalu $[0,2\pi]$ prolazi kroz sljedeće tačke $(0,\frac{\pi}{2})$, $(\frac{\pi}{2},\frac{\pi}{4})$ $(\pi,0)$, $(\frac{3\pi}{2},-\frac{\pi}{4})$, $(2\pi,-\frac{\pi}{2})$. Jednačina prave kroz dvije tačke je

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \quad \xRightarrow{\text{ako posmatramo } \overset{x_1\ y_1}{(0,\frac{\pi}{2})} \text{ i } \overset{x_2\ y_2}{(\pi,0)}} \quad \frac{x-0}{\pi}=\frac{y-\frac{\pi}{2}}{-\frac{\pi}{2}}$$

$$\Rightarrow \quad y-\frac{\pi}{2}=\frac{x}{\pi}\cdot\left(-\frac{\pi}{2}\right) \Rightarrow y-\frac{\pi}{2}=-\frac{x}{2} \Rightarrow y=\frac{\pi-x}{2}.$$

Furijeov red na intervalu $[a,b]$ je oblika

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$$

$$\frac{2n\pi x}{2\pi}=nx$$

gdje se Furijeovi koeficijenti računaju po formuli

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx, \qquad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,$$

$$b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi}{b-a}\,dx$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\,dx=\frac{1}{2\pi}\left(\pi x\Big|_0^{2\pi}-\frac{1}{2}x^2\Big|_0^{2\pi}\right)=\frac{1}{2\pi}(2\pi^2-2\pi^2)=0$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\cos nx\,dx=\frac{1}{2\pi}\int_0^{2\pi}(\pi-x)\cos nx\,dx=\begin{vmatrix} u=\pi-x & dv=\cos nx\,dx \\ du=-dx & v=\frac{1}{n}\sin nx\end{vmatrix}=$$

$$= \frac{1}{2\pi}\left(\frac{1}{n}(\pi - x)\sin nx\Big|_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right) = \underbrace{-\frac{1}{2n\pi}\sin 2n\pi}_{=0} + \frac{1}{2n\pi}\cdot\frac{-1}{n}\cos nx\Big|_0^{2\pi} =$$

$$= -\frac{1}{2n^2\pi}(\cos 2n\pi - 1) = \frac{1}{2n^2\pi}(1 - \cos 2n\pi) = \frac{1}{2n^2\pi}(1-1) = 0$$

Sa grafika date f-je možemo primjetiti da je f-ja simetrična u odnosu na koordinatni početak tj. da je neparna, pa je $a_0 = 0$ i $a_n = 0$ $\forall n \in \mathbb{N}$.

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin nx\,dx = \begin{vmatrix} u = \pi - x & dv = \sin nx\,dx \\ du = -dx & v = -\frac{1}{n}\cos nx \end{vmatrix}$$

$$= \frac{1}{2\pi}\left(-\frac{1}{n}(\pi - x)\cos nx\Big|_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\cos nx\,dx\right) = -\frac{1}{2n\pi}(-\pi\cos 2n\pi - \pi) -$$

$$- \frac{1}{2n\pi}\sin nx\Big|_0^{2\pi} = \frac{1}{2n}(\cos 2n\pi + 1) = \frac{1}{2n}\cdot 2 = \frac{1}{n}$$

Prema tome

$$f(x) \sim \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Ako za x uzmemo $\frac{\pi}{2}$ imamo

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi}{2}n = \sum_{k=1}^{\infty}\Big(\frac{1}{4k-1}\sin(4k-1)\frac{\pi}{2} + \frac{1}{4k-2}\underbrace{\sin(4k-2)\frac{\pi}{2}}_{\sin(2k-1)\pi = 0} +$$

$$+ \frac{1}{4k-3}\sin(4k-3)\frac{\pi}{2} + \frac{1}{4k}\underbrace{\sin 4k\frac{\pi}{2}}_{=\sin 2k\pi = 0}\Big) =$$

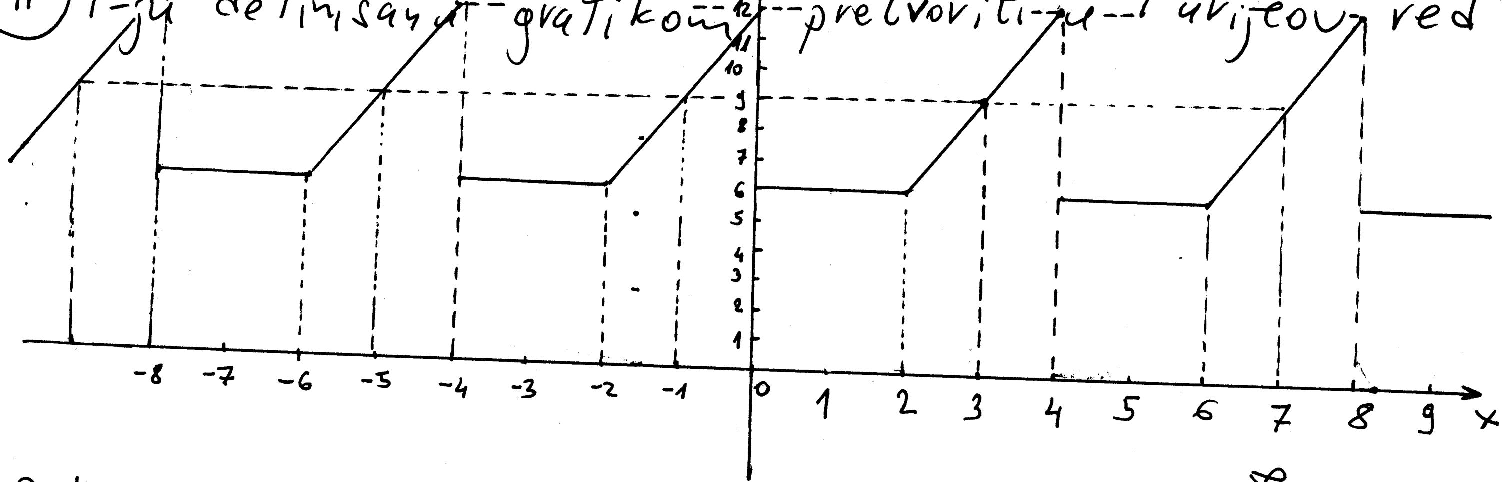
$$= \sum_{k=1}^{\infty}\frac{1}{4k-1}\sin\underbrace{(4k-1)\frac{\pi}{2}}_{=3,7,11,15} + \sum_{k=1}^{\infty}\frac{1}{4k-3}\sin\underbrace{(4k-3)\frac{\pi}{2}}_{=1,5,9,13}$$

$$= -1 \qquad\qquad = 1$$

$$= \sum_{k=1}^{\infty}\left(-\frac{1}{4k-1} + \frac{1}{4k-3}\right) = \sum_{k=1}^{\infty}\frac{4k-1-4k+3}{(4k-1)(4k-3)} = \sum_{k=1}^{\infty}\frac{2}{(4k-1)(4k-3)}$$

Kako je $f(\frac{\pi}{2}) = \frac{\pi}{4}$ to je $\displaystyle\sum_{n=1}^{\infty}\frac{2}{(4n-1)(4n-3)} = \frac{\pi}{4}$.

(#) F-ja definisana grafikom pretvoriti u Furijeov red



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$.

Rj. Prvo primjetimo da je data f-ja periodična perioda 4, što znači da je možemo pretvoriti u Furijeov red i to dovoljno ju je pretvoriti u Furijeov red na intervalu $(0,4)$.

Data f-ja na intervalu $(0,4)$ je definisana na sljedeći način $f(x) = \begin{cases} 6, & x \in (0,2] \\ 3x, & x \in (2,4) \end{cases}$.

Furijerov red na proizvoljnom intervalu $[a,b]$ izgleda

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$$

a Furijerovi koeficijenti računaju po formulama

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx, \qquad a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx, \quad n=1,2,\dots$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx, \quad n=1,2,\dots$$

Što znači Furijerov red na intervalu $[0,4)$ je

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right)$$

Izračunajmo Furijeove koeficijente

$$a_0 = \frac{1}{2}\int_0^4 f(x)\,dx = \frac{1}{2}\int_0^2 6\,dx + \frac{1}{2}\int_2^4 3x\,dx = 3x\Big|_0^2 + \frac{3}{2}\cdot\frac{1}{2}\overbrace{x^2\Big|_2^4}^{16-4} = 6 + \frac{3}{\cancel{4}_1}\cancel{12}^{3} = 15$$

$$a_n = \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_0^2 6\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_2^4 3x\cos\frac{n\pi x}{2}\,dx = \left| \begin{matrix} u=x & dv=\cos\frac{n\pi x}{2}dx \\ du=dx & v=\frac{2}{n\pi}\sin\frac{n\pi x}{2} \end{matrix} \right.$$

$$=\underbrace{3\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^2}_{0-0}+\frac{3}{2}\left[\underbrace{\frac{2}{n\pi}x\sin\frac{n\pi x}{2}\Big|_2^4}_{0-0}-\frac{2}{n\pi}\int_2^4\sin\frac{n\pi x}{2}dx\right]=$$

$$=-\frac{3}{n\pi}\int_2^4\sin\frac{n\pi x}{2}dx=\frac{3}{n\pi}\cdot\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_2^4=\frac{6}{n^2\pi^2}(1-\cos n\pi),\quad n\neq 0$$

Odavde vidimo $a_n=\begin{cases}0, & n \text{ parno}\\ \frac{12}{n^2\pi^2}, & n \text{ neparno}\end{cases}$ $n\in\mathbb{N}$

$$b_n=\frac{1}{2}\int_0^4 f(x)\sin\frac{n\pi x}{2}dx=\frac{1}{2}\int_0^2 6\sin\frac{n\pi x}{2}dx+\frac{1}{2}\int_2^4 3x\sin\frac{n\pi x}{2}dx=\left|\begin{matrix}u=x & dv=\sin\frac{n\pi x}{2}dx\\ du=dx & v=-\frac{2}{n\pi}\cdot\cos\frac{n\pi x}{2}\end{matrix}\right|$$

$$=3\left(-\frac{2}{n\pi}\right)\cos\frac{n\pi x}{2}\Big|_0^2+\frac{3}{2}\left[-\frac{2}{n\pi}x\cos\frac{n\pi x}{2}\Big|_2^4+\frac{2}{n\pi}\int_2^4\cos\frac{n\pi x}{2}dx\right]=$$

$$=\left(-\frac{6}{n\pi}\right)(\cos n\pi-1)+\frac{3}{2}\left[\left(-\frac{2}{n\pi}\right)(4\cos 2n\pi-2\cos n\pi)+\underbrace{\frac{2}{n\pi}\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_2^4}_{0-0}\right]$$

$$=\frac{6}{n\pi}(1-\cos n\pi)-\frac{3}{n\pi}(4-2\cos n\pi)=\frac{6}{n\pi}(1-\cos n\pi)-\frac{6}{n\pi}(2-\cos n\pi)$$

$$=\frac{6}{n\pi}(1\underline{-\cos n\pi}-2\underline{+\cos n\pi})=-\frac{6}{n\pi}$$

Prema tome $f(x)\sim\frac{15}{2}+\sum_{n=1}^{\infty}\left(\frac{6}{n^2\pi^2}(1-\cos n\pi)\cos\frac{n\pi x}{2}+\left(-\frac{6}{n\pi}\right)\sin\frac{n\pi x}{2}\right)$

$$=\frac{15}{2}+\sum_{k=1}^{\infty}\left(\frac{12}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi x}{2}-\frac{6}{k\pi}\sin\frac{k\pi x}{2}\right)$$

$$f(x)\sim\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\frac{\pi}{2}x}{(2k-1)^2}-\frac{6}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\frac{\pi}{2}x}{k}$$

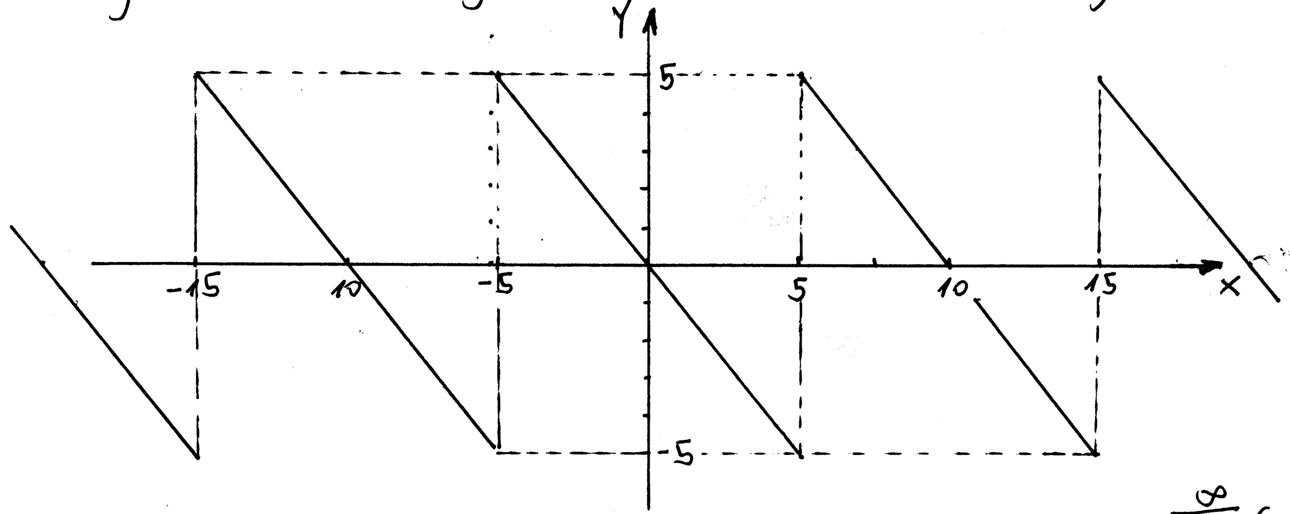
Za $x=2$ imamo

$$f(2)=\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\pi}{(2k-1)^2}-\underbrace{\frac{6}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\pi}{k}}_{=0}$$

$$6=\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)}{(2k-1)^2}\Rightarrow-\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=-\frac{3}{2}$$

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\left(-\frac{3}{2}\right)\left(-\frac{\pi^2}{12}\right)=\frac{\pi^2}{2\cdot 4}\qquad\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}$$ tražena suma

(#) Funkciju definisanu grafikom pretvoriti u Furijeov red.



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi}{50}$.

Rj. Primjetimo da je data f-ja periodična perioda 10. Prema tome dovoljno ju je pretvoriti u Furijeov red na proizvoljnom intervalu periodu 10. Pa posmatrajmo npr. interval $[-5, 5]$. F-ja na ovom intervalu ima oblik $f(x) = -x$.
Furijeov red f-je $f(x)$ na intervalu $[a,b]$ ima oblik:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$$

gdje su $a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$, $a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx$ i $b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$ $\quad n = 1, 2, \dots$

Furijeovi koeficijenti. U našem slučaju interval $[a,b]$ je $[-5, 5]$ pa je $b-a = 5+5 = 10$, $\frac{2}{10} = \frac{1}{5}$, $\frac{2n\pi x}{b-a} = \frac{2n\pi x}{10} = \frac{n\pi x}{5}$.

$$a_0 = \frac{1}{5}\int_{-5}^{5} (-x)\,dx = \frac{1}{5}(-1)\cdot\frac{1}{2}x^2\Big|_{-5}^{5} = 0 \qquad d\left(\frac{n\pi x}{5}\right) = \frac{n\pi}{5}dx$$

$$a_n = \frac{1}{5}\int_{-5}^{5} (-x)\cos\frac{n\pi x}{5}\,dx = \left|\begin{matrix} u = x & dv = \cos\frac{n\pi x}{5}dx \\ du = dx & v = \frac{5}{n\pi}\sin\frac{n\pi x}{5} \end{matrix}\right| =$$

$$= -\frac{1}{5}\left(\overbrace{\frac{5}{n\pi}x\sin\frac{n\pi x}{5}\Big|_{-5}^{5}}^{=0} - \frac{5}{n\pi}\int_{-5}^{5}\sin\frac{n\pi x}{5}\,dx\right) = \frac{1}{n\pi}\left(-\frac{5}{n\pi}\right)\cos\frac{n\pi x}{5}\Big|_{-5}^{5} = 0$$

$$b_n = \frac{1}{5}\int_{-5}^{5}(-x)\sin\frac{n\pi x}{5}\,dx = \left| \begin{matrix} u = x & dv = \sin\frac{n\pi x}{5}\,dx \\ du = dx & v = \frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right) \end{matrix} \right| =$$

$$= -\frac{1}{5}\left(\frac{-5}{n\pi}\,x\cos\frac{n\pi x}{5}\Big|_{-5}^{5} + \frac{5}{n\pi}\int_{-5}^{5}\cos\frac{n\pi x}{5}\,dx\right) =$$

$$= \frac{1}{n\pi}\left(5\cos n\pi - (-5)\cos n\pi\right) - \frac{1}{n\pi}\cdot\underbrace{\frac{5}{n\pi}\sin\frac{n\pi x}{5}\Big|_{-5}^{5}}_{=0} =$$

$$= \frac{10}{n\pi}\cos n\pi = \frac{10}{n\pi}(-1)^n$$

Prema tome

$$-x \sim \sum_{n=1}^{\infty}\frac{10}{n\pi}(-1)^n\sin\frac{n\pi x}{5} \quad \text{za } \forall x\in[-5,5]$$

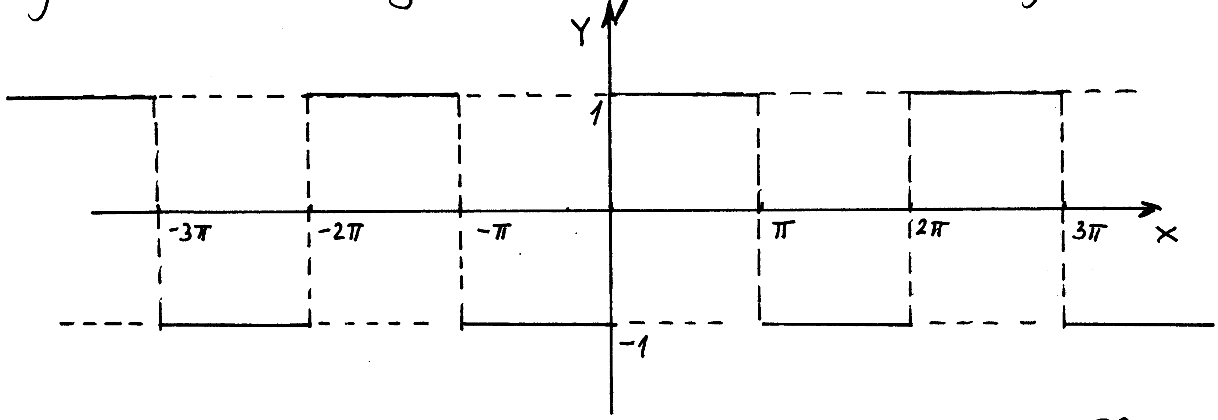
tj. $$-x = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{5} \quad \text{za } \forall x\in[-5,5]$$

Ako za x uzmemo $x=\frac{1}{10}$ imamo:

$$-\frac{1}{10} = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi\cdot\frac{1}{10}}{5}$$

tj. $$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{50} = -\frac{\pi}{100} \quad \text{tražena suma}$$

F-ju definisanu grafikom pretvoriti u Furijer-ov red.



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Rj: Primjetimo da je data f-ja periodična, perioda 2π, pa je možemo pretvoriti u Furijer-ov red. Kada je x-osa data u radijanima Furijer-ov red izgleda $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

gdje se Furijer-ovi koeficijenti računaju u obliku (za 2π per. f-ju)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx + \frac{1}{2}\int_{0}^{\pi} 1\,dx = \left(-\frac{1}{\pi}\right)x\Big|_{-\pi}^{0} + \frac{1}{\pi}x\Big|_{0}^{\pi} = -1+1=0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx = \left(-\frac{1}{\pi}\right)\frac{1}{n}\sin nx\Big|_{-\pi}^{0} + \frac{1}{\pi}\cdot\frac{1}{n}\sin nx\Big|_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} b_n \sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx = \left(-\frac{1}{\pi}\right)\left(-\frac{1}{n}\right)\cos nx\Big|_{-\pi}^{0} + \frac{1}{\pi}\left(-\frac{1}{n}\right)\cos nx\Big|_{0}^{\pi}$$

$$= \frac{1}{n\pi}(1-\cos n\pi) - \frac{1}{n\pi}(\cos n\pi - 1) = \frac{2}{n\pi}(1-\cos n\pi)$$

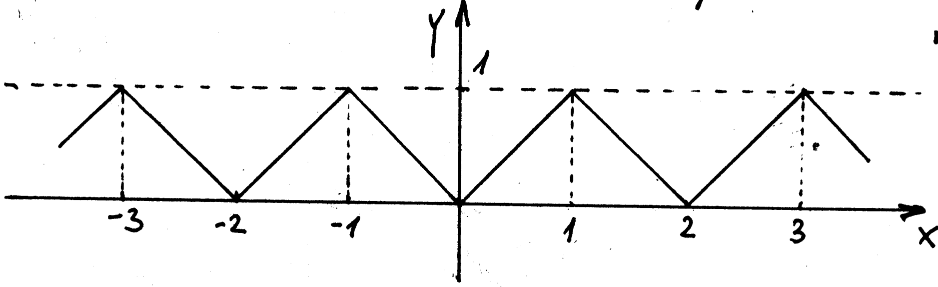
$$1-\cos n\pi = 1-(-1)^n = \begin{cases} 0, & n=2k \\ 2, & n=2k+1 \end{cases} \quad k=0,1,2,\ldots$$

$$\boxed{\sin(2k+1)\frac{\pi}{2} = (-1)^k}$$

$f(x) \sim \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}$ traženi Furijerov red

$f\left(\frac{\pi}{2}\right)=1 \quad \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\frac{\pi}{2}}{2k+1} = 1 \implies \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}$ tražena suma

(#) F-ju definisanu grafikom razviti u Fourierov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.



Rj: Sa grafika možemo primijetiti da je f-ja parna i periodična perioda 2. F-ju je dovoljno razviti u Fourierov red u intervalu $[-1,1]$, pa kako je f-ja parna imaćemo da su $b_n=0$ $\forall n$.
Ako f-ju označim sa $f(x)$ na intervalu $[-1,1]$ imamo
$$f(x)=\begin{cases} x, & 0\le x\le 1\\ -x, & -1\le x\le 0\end{cases}$$

Ako je $f(x)$ integrabilna f-ja na intervalu $[-l,l]$ Fourierove koeficijente računamo po formulama

$$a_0=\frac{1}{l}\int_{-l}^{l} f(x)\,dx,\quad a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,\quad b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Fourierov red f-je $f(x)$ je tad oblika:
$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

F-ja je parna:
$$a_0=\frac{2}{l}\int_0^l f(x)\,dx=2\int_0^1 x\,dx=2\cdot\frac{1}{2}x^2\Big|_0^1=1$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=2\int_0^1 x\cos n\pi x\,dx=\begin{vmatrix} u=x & dv=\cos n\pi x\,dx\\ du=dx & v=\frac{1}{n\pi}\sin n\pi x\end{vmatrix}=$$

$$=\underbrace{\frac{2}{n\pi}x\sin n\pi x\Big|_0^1}_{=0}-\frac{2}{n\pi}\int_0^1\sin n\pi x\,dx=-\frac{2}{n\pi}\cdot\frac{(-1)}{n\pi}\cos n\pi x\Big|_0^1=2\,\frac{\cos n\pi-\cos 0}{n^2\pi^2}$$

$$a_n=2\,\frac{(-1)^n-1}{n^2\pi^2},\quad b_n=0\ \forall n\ \Rightarrow\ f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{\pi^2}\cdot\frac{-4}{(2n-1)^2}\cos(2n+1)\pi x$$

$$f(x)=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n+1)^2}$$ razlaganje f-je u Fourierov red

Za $x=0$, $f(0)=0$ $\Rightarrow$ $\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$